MATH2050C Assignment 7

Deadline: March 11, 2025.

Hand in: 3.7 no. 3c, 10, 15, 16; 4.1 no. 11b, 12d, 15; 4.2 no. 1c, 2b.

Section 3.7 no. 3ac, 7, 10, 11, 12, 15, 16;

Section 4.1 no. 7, 8, 9bd, 10b, 11b, 12bd, 15;

Section 4.2 no. 1bc, 2bd.

Supplementary Problems

1. An infinite series $\sum_n x_n$ is called **absolutely convergent** if $\sum_n |x_n|$ is convergent. Show that an absolutely convergent infinite series is convergent but the convergence of $\sum_n x_n$ does not necessarily imply the convergence of $\sum_n |x_n|$.

2. Prove by the Limit Theorem (see next page) that $\lim_{x\to c} p(x) = p(c)$ for any polynomial p and real number c.

3. Let f be a function on A and c a cluster point of A. Show that $\lim_{x\to c} |f(x)| = |L|$ whenever $\lim_{x\to c} f(x) = L$.

4. Let f be a non-negative function on A and c a cluster point of A. Suppose that $\lim_{x\to c} f(x) = L$ for some L. Show that $\lim_{x\to c} \sqrt{f(x)} = \sqrt{L}$. Suggestion: Consider $L > 0$ and $L = 0$ separately.

See next page

Basic Examples of Infinite Series

Let $\sum_{n=1}^{\infty} x_n$ be an infinite series. Its n-th partial sum s_n is given by $\sum_{k=1}^{n} x_k$. An infinite series $\sum_{n=1}^{\infty} x_n$ is called **convergent/divergent** if the sequence $\{s_n\}$ is convergent/divergent. When an infinite series converges, we use $\sum_{n=1}^{\infty} x_n$ to denote the limit $\lim_{n\to\infty} s_n$. Thus, the notation $\sum_{n=1}^{\infty} x_n$ has two meanings; first it is the notation for an infinite series, and second, it is the ultimate sum of the infinite series (provided it converges).

Sometimes, $\sum_{n=1}^{\infty} x_n$ is replaced by the simpler $\sum_{n}^{\infty} x_n$ or $\sum_n x_n$.

Basic examples of infinite series. You should know the proofs behind them.

- $$\sum_{n=1}^{\infty} (-1)^{n+1}$$
  is divergent.

- For $\alpha \in (0,1)$,
  $$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \ .$$

- $$\sum_{n=1}^{\infty} \frac{1}{n^a}$$
  is convergent if and only if $a > 1$.

- The alternating harmonic series
  $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$
  is convergent.

The following comparison theorem is one of the most common test for convergence/divergence for infinite series. We will discuss infinite series at length in MATH2060.

Comparison Theorem Let $0 \le x_n \le y_n$ for all n. Then (a) $\sum_{n=1}^{\infty} y_n$ converges implies $\sum_{n=1}^{\infty} x_n$ converges; and (b) $\sum_{n=1}^{\infty} x_n$ diverges implies $\sum_{n=1}^{\infty} y_n$ diverges.

The Limit Theorem for Functions

Theorem 7.1 (**Limit Theorem**) Let c be a cluster point of A and f, g functions on A satisfying $f(x) \to L, g(x) \to M$ as $x \to c$ respectively. Then

1. $\lim_{x\to c}(\alpha f + \beta g) = \alpha L + \beta M$.

2. $\lim_{x\to c}(fg)(x) = LM$.
3. $\lim_{x\to c}\left(\dfrac{f}{g}\right)(x) = \dfrac{L}{M}$ provided $M \neq 0$.

By induction, (1) and (2) of this theorem which hold for the sum or the product of two terms can be extended to the sum or the product of finitely many terms.

In our textbook this theorem is proved by the Sequential Criterion. In class we proved it by using the ε-δ definition. Here we repeat it for the product rule. Indeed, we have

$$\begin{aligned} |(fg)(x) - LM| &= |f(x)g(x) - LM| \\ &= |(f(x) - L)g(x) + L(g(x) - M)| \\ &\leq |g(x)||f(x) - L| + |L||g(x) - M| \ . \end{aligned}$$

As $g(x) \to M$, for $\varepsilon = 1$, there is some δ_0 such that $|g(x) - M| < 1$ for $x \in A, 0 < |x - c| < \delta_0$. So $|g(x)| \leq |M| + 1$ there. We have

$$|(fg)(x) - LM| \leq (|M| + 1)|f(x) - L| + |L||g(x) - M| \ ,$$

whenever $0 < |x - c| < \delta_0$. Now given $\varepsilon > 0$, as $f(x) \to L$ and $g(x) \to M$, there are δ_1, δ_2 such that $|f(x) - L| < \varepsilon/(|L| + |M| + 1)$ for $0 < |x - c| < \delta_1$ and $|g(x) - M| < \varepsilon/(|L| + |M| + 1)$ for $0 < |x - c| < \delta_2$. It follows that for $x, 0 < |x - c| < \delta$ where $\delta = \min\{\delta_0, \delta_1, \delta_2\}$,

$$|(fg)(x) - LM| < (|M| + 1)\frac{\varepsilon}{|L| + |M| + 1} + |L|\frac{\varepsilon}{|L| + |M| + 1} = \varepsilon \ ,$$

done.

Theorem 7.2 (**Sequential Criterion**) The following statements are equivalent:

(a) $\lim_{x\to c} f(x) = L$;

(b) For any sequence $\{x_n\}, x_n \neq c, x_n \to c, f(x_n) \to L$ as $n \to \infty$.

Here are some applications of the Sequential Criterion.

Example 7.1 Let p be a polynomial. For $c \in \mathbb{R}$, $\lim_{x\to c} p(x) = p(c)$. A polynomial is well-defined everywhere on the real line. It is of the form $a_0 + a_1x + \cdots + a_nx^n$ for some n. It was shown in Chapter 3 that $\lim_{n\to\infty} p(x_n) = p(c)$ for any sequence $x_n \to c$. By the Sequential Criterion $\lim_{x\to c} p(x) = p(c)$. You may also prove the same result using the Limit Theorem for functions.

Example 7.2 Let p, q be two polynomials. For c satisfying $q(c) \neq 0$, $\lim_{x\to c} p(x)/q(x) = p(c)/q(c)$. This conclusion comes from Example 7.1, Sequential Criterion and the quotient rule for sequences.

Example 7.3 In Chapter 3 it was shown that the function $x^{p/q}, p, q \in \mathbb{N}$, is well-defined for $x \in [0, \infty)$. And for any sequence $x_n \to c \in [0, \infty)$, $x_n^{p/q} \to c^{p/q}$. Immediately it follows from the Sequential Criterion that $\lim_{x\to c} x^{p/q} = c^{p/q}$.

The Divergence Criteria follows directly from the Sequential Criterion.

Proposition 7.3 (Divergence Criteria) $\lim_{x\to c} f(x)$ does not exist in either one of the following two cases:

(a) There is $x_n \in A, x_n \neq c, x_n \to c$, such that $\{f(x_n)\}$ is unbounded;

(b) There are $x_n, y_n \in A$ not equal to c and $x_n, y_n \to c$ such that $f(x_n) \to L_1, f(y_n) \to L_2$ with $L_1 \neq L_2$.

Example 7.4 $\lim_{x\to 0} 1/x^p, p \in \mathbb{N}$, does not exist. Consider the sequence $x_n = 1/n \to 0$, we have $1/x_n^p = n^p \to \infty$. By the Divergence Criterion (a) this limit does not exist.

Example 7.5 $\lim_{x\to 0} \sin 1/x$ does not exist. Consider two sequences $x_n = 1/2\pi n$ and $y_n = 1/(2\pi n + \pi/2)$. Then $\sin 1/x_n = \sin 2\pi n = 0$ and $\sin 1/y_n = \sin(2\pi n + \pi/2) = 1$ for all n. Hence $L_1 = 0$ and $L_2 = 1$. By Divergence Criterion (b), the limit does not exist.

The third consequence of the Sequential Criterion is the Squeeze Theorem.

Theorem 7.4 (Squeeze Theorem) Suppose that $f(x) \leq g(x) \leq h(x),\ x \in A$, and

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L\ .$$

Then $\lim_{x\to c} g(x) = L$.

This theorem follows from the corresponding theorem for sequences and the Sequential Criterion.

Example 7.6 $\lim_{x\to 0} \sin x = 0$. Using the estimate $0 \leq \sin x \leq x$ for $x \in [0,1]$ and the fact that the sine function is odd, $-|x| \leq \sin x \leq |x|, x \in [-1,1]$. A direct application of the Squeeze Theorem gives the desired limit.

Example 7.7 $\lim_{x\to 0} \sin x/x = 1$. This follows readily from the estimate $x - x^3/6 \leq \sin x \leq x, x \in [0,1]$ (see below). Derive both sides by x, we have $1 - x/6 \leq \sin x/x \leq 1$. Since $\sin x/x$ is even, this estimate holds on $[-1,0) \cup (0,1]$. By Squeeze Theorem $\lim_{x\to 0} \sin x/x = 1$.

The sine and cosine functions will come more frequently in our later development. We will discuss them in next week.